FUNDAMENTALS OF ELECTROMAGNETIC FIELDS – January 22nd, 2018

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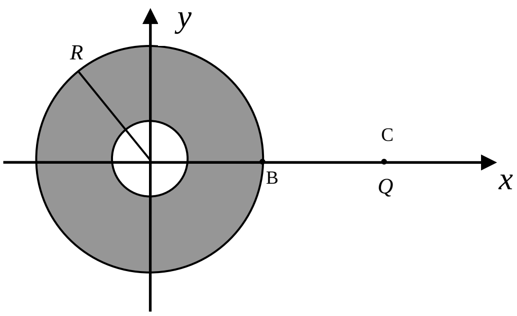
Problem 1 - A uniform charge density $\rho = 10^{-10}$ C/m^3 is embedded in a hollow dielectric sphere with radius R = 4 m (grey part in the figure). A point charge, with value $Q = 3 \cdot 10^{-9}$ C, is fixed in point C(2R,0). Another charge q, free to move and with value -Q, is placed at the sphere center. Determine the value of another point charge Q_B to be placed in B(R,0) such that q is in equilibrium.



Solution

The electric field in (0,0) due to the charged hollow sphere is zero, as stated by the Gauss' theorem:

$$\int_S \vec{D} \cdot d\vec{S} = \int_V \rho \, dV$$

As q is negative and Q positive, the latter will tend to attract q; therefore, q will be subject to the following force:

$$\vec{F}_C = q\vec{E}_Q(0,0) = -|Q|\vec{E}_Q(0,0) = -|Q|\left[-\frac{|Q|}{4\pi\varepsilon_0(2R)^2}\vec{\mu}_x\right] = \frac{|Q|^2}{16\pi\varepsilon_0 R^2}\vec{\mu}_x$$

In order to counterbalance the effect of Q in C, the charge to be placed in B must be negative so as to repulse q, which will be subject to:

$$\vec{F}_B = q\vec{E}_{Q_B}(0,0) = -|Q|\vec{E}_{Q_B}(0,0) = -|Q|\left[\frac{|Q_B|}{4\pi\varepsilon_0(R)^2}\vec{\mu}_x\right] = -\frac{|Q||Q_B|}{4\pi\varepsilon_0 R^2}\vec{\mu}_x$$

The equilibrium for q is reached if:

$$\vec{F} = \vec{F}_B + \vec{F}_C = 0$$

which yields:

$$\frac{|Q||Q_B|}{4\pi\varepsilon_0 R^2} = \frac{|Q|^2}{16\pi\varepsilon_0 R^2}$$

Therefore:

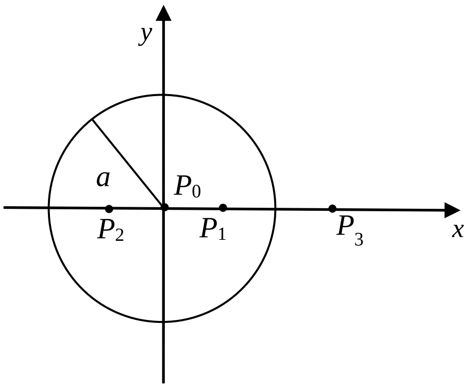
$$|Q_B| = \frac{|Q|}{4} = 7.5 \cdot 10^{-10} \text{ C} \quad \rightarrow \quad Q_B = -7.5 \cdot 10^{-10} \text{ C}$$

Problem 2 - Let us consider the following distribution for the current density flowing inside a cylindrical wire (see the wire section in the figure with radius a = 0.5 cm):

$$\vec{J} = \begin{cases} -\frac{2\rho}{a}\vec{a}_z & \rho \le a \\ 0 & \rho > a \end{cases} \quad \text{A/m}^2$$

where ρ is the distance from the wire axis.

Calculate the magnetic field (full vector) in the following points P_0 (0,0,0), P_1(a/2,0,0), P_2(-a/2,0,0) e P_3(2a,0,0).



Solution

The solution is easily achieved using the Ampère's theorem, i.e.:

$$\int_l \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{S}$$

For point P_0, $\vec{H}(P_0) = 0$ because no current is included in the circle l.

For point P_1, Ampère's theorem becomes:

$$\left|\vec{H}\right| 2\pi\rho = \int_0^\rho \frac{2r}{a} 2\pi r\, dr = \frac{4\pi}{3a}\rho^3 \quad \Rightarrow \quad \vec{H}(P_1) = -\frac{2\rho^2}{3a}\vec{a}_y\Big|_{P_1} = -8.3\cdot 10^{-4}\vec{a}_y \ \text{A/m}$$

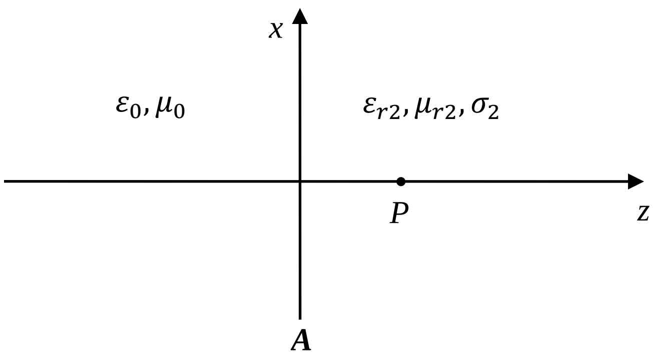
For point P_2, the solution is the same as for point P1, but the magnetic field vector changes direction:

$$\vec{H}(P_1) = 8.3\cdot 10^{-4}\vec{a}_y \ \text{A/m}$$

Finally, for point P_3:

$$\left|\vec{H}\right| 2\pi\rho = \int_0^a \frac{2r}{a} 2\pi r\, dr = \frac{4\pi}{3}a^2 \quad \Rightarrow \quad \vec{H} = -\frac{2a^2}{3\rho}\vec{a}_y\Big|_{P_3} = -0.0017\vec{a}_y \ \text{A/m}$$

Problem 3 - A uniform plane wave (frequency f = 200 MHz) propagates in free space and impinges on a dielectric material ($\varepsilon_{r2} = 4, \mu_{r2} = 1, \sigma_2 = 4.5 \cdot 10^{-2}$ S/m). The electric field of the incident wave at (0,0,0) is $\vec{E}_i(0,0,0) = -j\vec{a}_x$ V/m. Calculate the electric field in $P(0,0,\lambda_2)$ and the power density carried by the wave in $P(0,0,\lambda_2)$ (λ_2 is the wavelength in the second medium).



Solution

First, let us calculate the intrinsic impedance for the two media. For the first one:

$$\eta_1 = \eta_0 \approx 377\ \Omega$$

For the second one, the loss tangent is $\boldsymbol{\sigma/\omega\varepsilon \approx 1}$. Therefore no accurate approximations are possible:

$$\eta_2 = \sqrt{\frac{j\omega\mu_2}{\sigma + j\omega\varepsilon_2}} = 146 + j61\,\Omega$$

The reflection coefficient is:

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = -0.423 + j0.166$$

Also, let us calculate the propagation constant for the second medium:

$$\gamma_2 = \alpha_2 + j\beta_2 = \sqrt{(j\omega\mu_2)(\sigma + j\omega\varepsilon_2)} = 3.85 + j9.23 \quad \text{1/m}$$

The wavelength is obtained from β_2 as:

$$\lambda_2 = \frac{2\pi}{\beta_2} \approx 0.68 \quad \text{m}$$

The field transmitted into the second medium is:

$$\vec{E}_2(0,0,0) = \vec{E}_i(0,0,0)(1+\Gamma) = (0.166 - j0.577)\vec{\mu}_x \quad \text{V/m}$$

The electric field in P is therefore:

$$\vec{E}_2(P) = \vec{E}_2(0,0,0)e^{-\gamma_2\lambda_2} = (0.012 - j0.042)\vec{\mu}_x \quad \text{V/m}$$

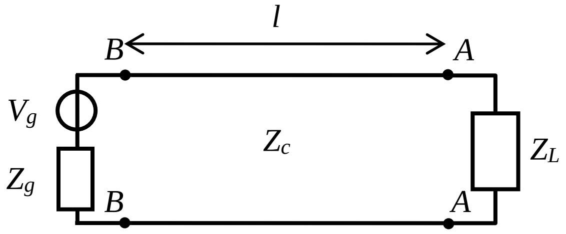
Finally, the power density is:

$$S(P) = \frac{1}{2} \frac{\left|\vec{E}_2(P)\right|^2}{|\eta_2|} \cos(\angle \eta_2) = 5.5 \ \ \mu\text{W/m}^2$$

<u>Problem 4</u> **-** A source with voltage V_g = 50 V and internal impedance Z_g = 100 Ω is connected to a load Z_L = 150 Ω by a transmission line with characteristic impedance Z_C = 50 Ω. The line length is l = 4 m and the frequency is f = 300 MHz.

Calculate:

a) The power absorbed by the load
b) The voltage at the beginning of the line (section BB below), V_B
c) The trend of V_B in time



Solution

a) The wavelength is:

$$\lambda = \lambda_0 = c/f = 1 \text{ m}$$

The reflection coefficient at section AA is:

$$\Gamma_A = \frac{Z_L - Z_C}{Z_L + Z_C} = 0.5$$

The reflection coefficient at section BB is:

$$\Gamma_B = \Gamma_A e^{-j2\beta l} = \Gamma_A e^{-j2\left(\frac{2\pi}{\lambda}\right)4\lambda} = \Gamma_A e^{-j16\pi} = 0.5$$

Therefore, the input impedance is:

$$Z_B = Z_C \frac{1+\Gamma_B}{1+\Gamma_B} = 150 \quad \Omega$$

The reflection coefficient for the source is:

$$\Gamma_g = \frac{Z_B - Z_g}{Z_B + Z_g} = 0.2$$

Therefore, the power crossing section BB, i.e. reaching the load is:

$$P_L = P_{AV}(1 - |\Gamma_g|^2) = 3 \text{ W}$$

b) The voltage at the beginning of the line is:

$$V_B = V_g \frac{Z_B}{Z_B + Z_g} = 30 \text{ V}$$

c) The trend of V_B in time is given by:

$$v_B(t) = \mathrm{Re}\left[V_B \, e^{j\omega t}\right] = 30\cos(\omega t) \quad \text{V}$$