Electromagnetics and Signal Processing for Spaceborne Applications
July 19th, 2023

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SURNAME AND NAME ______________________________

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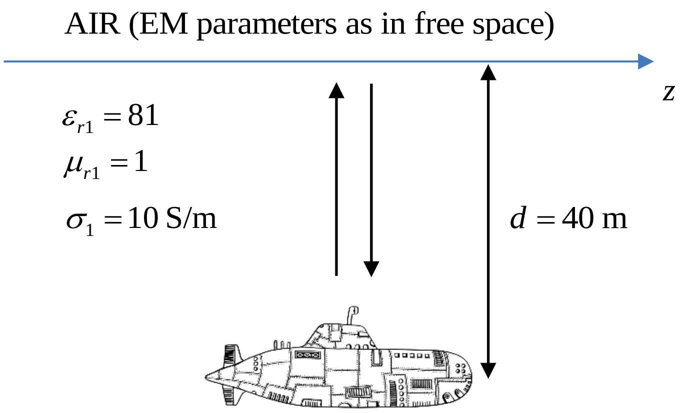
Problem 1

A submarine transmits electromagnetic signals towards the sea surface to keep track of its depth. Assuming that the submarine antenna emits plane waves with electric field $|\vec{\boldsymbol{E}}_{\boldsymbol{out}}| = \mathbf{5\ V/m}$ at frequency f = 1 kHz, and that its depth is d = 40 m, calculate:

1) The wavelength underwater.
2) The propagation velocity underwater.
3) The power density reaching back the submarine after the reflection on the sea surface.



Solution

1) First we need to characterize the electromagnetically the first medium (sea water). In this case, the loss tangent is $\frac{\boldsymbol{\sigma}}{\boldsymbol{\omega\varepsilon}}$ = 2.2×10^6 >> 1. Therefore the second medium can be well approximated as a good conductor. Accordingly, the attenuation and propagation constants are:

$\alpha_1 = \beta_1 = \sqrt{\pi f \mu_1 \sigma_1} = 0.1987$ 1/m

As for the intrinsic impedance, we obtain:

$$\eta_1 = \sqrt{\frac{\pi f \mu_1}{\sigma_1}}(1+j) = 0.0199(1+j)\ \Omega$$

The wavelength is:

$$\lambda_1 = \frac{2\pi}{\beta_1} = 31.62 \text{ m}$$

2) The propagation velocity is:

$$v_1 = \frac{\omega}{\beta_1} = 3.162\times10^4 \text{ m/s}$$

3) For the second medium (air/free space), $\eta_2 = 377\ \Omega$. The reflection coefficient is therefore:

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \approx 1$$

The power density emitted by the submarine antenna is:

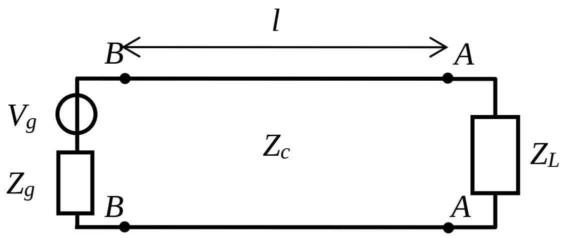
$$S_{out} = \frac{1}{2}\frac{\left|\vec{E}_{out}\right|^2}{|\eta_1|}\cos(\measuredangle\eta_1) = 314.6 \text{ W/m}^2$$

The power density reaching the submarine after reflection is: 4.9 pW/m^2

Problem 2

A source with voltage V_g = 10 V and internal impedance Z_g = 50 Ω is connected to a load Z_L = -j10 Ω by a transmission line with characteristic impedance Z_C = 100 Ω. The line length is l = 6.25 m and the frequency is f = 300 MHz.
For this circuit:

1) What kind of load is Z_L?
2) Calculate the power absorbed by the load.
3) Write the expression of the temporal trend of the voltage at the beginning of the line, V_{BB} (section BB).
4) Assuming to change the load, determine the value of Z_L to maximize P_{BB}, the power transferred beyond section BB, and the value of P_{BB} in this case.



Solution

1) The load, being imaginary (negative), corresponds to a capacitor.

2) As the load is imaginary, no power will be absorbed by Z_L.

3) The wavelength is:

$\lambda = \lambda_0 = c/f = 1$ m

The length of the line l corresponds to:

$\lambda = 6\lambda + \lambda/4$

Therefore, the input impedance is simply given by:

$$Z_{BB} = \frac{Z_C^2}{Z_L} = j1000 \ \Omega$$

The voltage at the beginning of the line is:

$$V_{BB} = V_g \frac{Z_{BB}}{Z_{BB} + Z_g} = 9.9751 + j0.4988 \text{ V} \rightarrow |V_{BB}| = 9.98 \ \mathbf{V/m} \text{ and } \measuredangle V_{BB} = 0.05 \text{ rad.}$$

Therefore:

$V_{BB}(t) = 9.98\cos(2\pi f t + 0.05)$ V.

4) To maximize the power transfer, Z_{BB} needs to be equal to Z_g:

$$Z_{BB} = \frac{Z_C^2}{Z_L} = Z_g = 50 \ \Omega \rightarrow Z_L = \frac{Z_C^2}{Z_g} = 200 \ \Omega$$

In this case, $\Gamma_g = 0$ and therefore $P_L = P_d = \dfrac{|V_g|^2}{8\text{Re}\{Z_g\}} = 0.25$ W.

Problem 3

A signal is modeled as:

$$s(t) = A \cdot cos(2\pi f_A t) + B \cdot cos(2\pi f_B t)$$

With $f_A = 100$ Hz and $f_B = 200$ Hz. The signal is observed for a total observation time To = 1 second.

1. Write the expressions of the Fourier Transform of the signal and draw the graph of its magnitude.
2. Discuss if it is possible to resolve the two sinusoids and measure their amplitudes A and B.
3. The signal is sampled with sampling rate fs = 300 Hz to produce a numerical sequence s_n. Write the expressions of the Fourier Transform of the numerical sequence s_n. and draw the graph of its magnitude. Can you resolve the two sinusoids?
4. Repeat point 3 assuming a sampling frequency fs = 250 Hz. Can you resolve the two sinusoids now?
5. Write a short pseudo-code to find the frequencies of the two sinusoids and measure their amplitudes A and B.

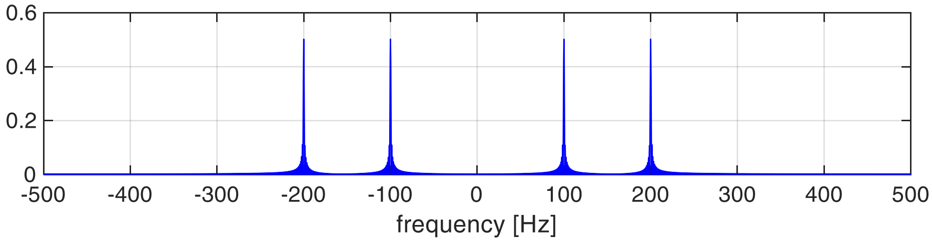
Solution

The Fourier Transform of the signal for an infinite observation time is:

$$S(f) = \frac{A}{2}\delta(f + f_A) + \frac{A}{2}\delta(f - f_A) + \frac{B}{2}\delta(f + f_B) + \frac{B}{2}\delta(f - f_B)$$

For a finite observation time the deltas are replaced by cardinal sines with amplitude T_o and width $1/T_o$:

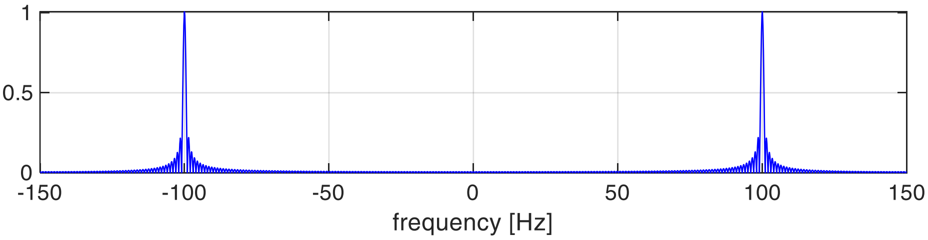
$$S(f) = \frac{AT_o}{2} sinc\big((f + f_A)T_o\big) + \frac{AT_o}{2} sinc\big((f - f_A)T_o\big) + \frac{BT_o}{2} sinc\big((f + f_B)T_o\big) + \frac{BT_o}{2} sinc\big((f - f_B)T_o\big)$$



Frequency resolution is 1/To = 1 Hz, so the sinusoids are well resolved and their amplitudes can be told based on the amplitudes of detected peaks ($\frac{AT_o}{2}$).

Sampling at 300 Hz results in the sinusoids at +- 200 Hz to be aliased, and mapped at -+100 Hz. Accordingly, the two sinusoids are overlapped and can no longer be distinguished:

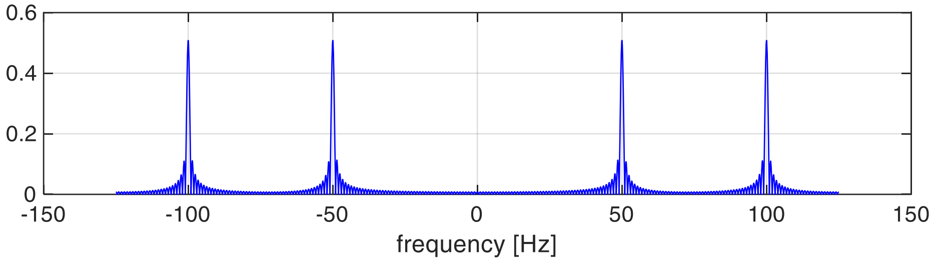
$$S(f) = \frac{(A + B)T_o}{2} sinc\big((f + f_A)T_o\big) + \frac{(A + B)T_o}{2} sinc\big((f - f_A)T_o\big)$$



Sampling at 300 Hz results in the sinusoids at +- 200 Hz to be aliased, and mapped at -+50 Hz. Accordingly, the two sinusoids do not overlap and can be distinguished:

$$S(f) = \frac{AT_o}{2} sinc\big((f + f_A)T_o\big) + \frac{AT_o}{2} sinc\big((f - f_A)T_o\big) + \frac{BT_o}{2} sinc\big((f + f_C)T_o\big) + \frac{BT_o}{2} sinc\big((f - f_c)T_o\big)$$

With $f_C = 50$ Hz.



Pseudo code:

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f = (-Nf/2:Nf/2-1)/Nf/dt;
S = fftshift(fft(s,Nf))*dt;
[PA,kA] = max(abs(S));
fA = f(kA)
A = 2*PA/To
% zeroing the samples corresponding to the first peak
kA1 = find(f==fA);
kA2 = find(f==-fA);
df = f(2)-f(1);
L = round(1/To/df); % Main lobe width in samples
S(kA1 +(-L:L)) = 0;
S(kA2 +(-L:L)) = 0;
% peak of the second sinusoid
[PB,kB] = max(abs(S));
fB = f(kB)
B = 2*PB/To
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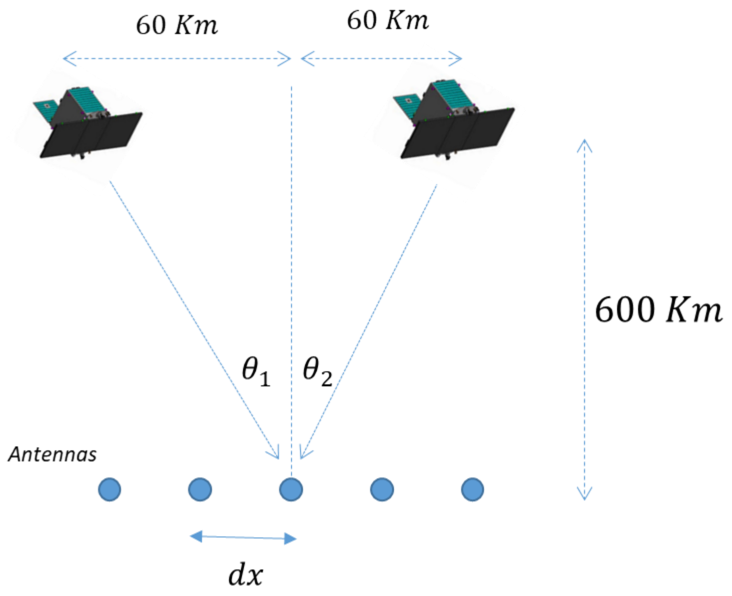
Problem 4

A small antenna array is used to capture the signals emitted by two satellites, as depicted in the figure below:



The array is formed by N=5 antennas and the separation between nearby antennas is dx = 0.5 m. The signals emitted from the two sources can be assumed to be monochromatic at the frequency f_0= 300 MHz.

1. Write the expression of the complex envelope of the signal received by each of the 5 antennas.
2. Discuss whether it is possible to resolve the angular position of the two satellites based on (the complex envelope of) the signals gathered at the 5 antennas.
3. Repeat point 2 under the assumption that the separation between nearby antennas is dx = 2 m.
4. Calculate the angular interval within which no ambiguities occur in the two cases dx = 0.5 m and dx = 2 m.
5. Write a short pseudo-code to find the angular position of the two satellites based on (the complex envelope of) the signals gathered at the 5 antennas.

Solution

The distances from antenna placed at (x,0) are:

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R1(x) = sqrt(H^2 + (x1-x).^2);
R2(x) = sqrt(H^2 + (x2-x).^2);
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Hence the field at x is:

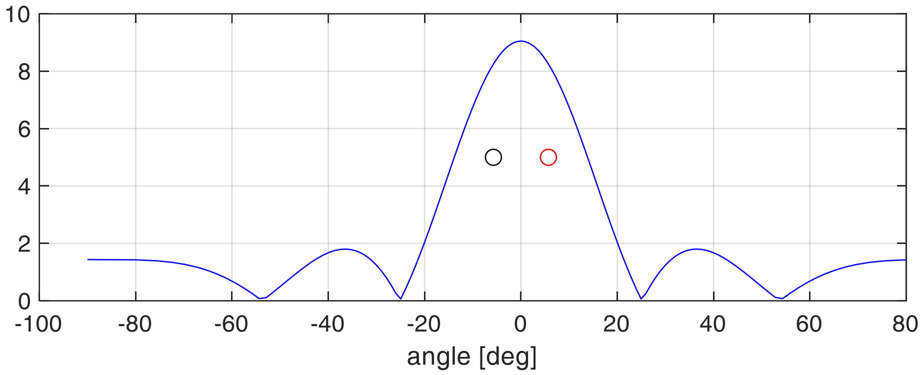
$$E(x) = \frac{1}{R_1(x)} e\left(-j\frac{2\pi}{\lambda}R_1(x)\right) + \frac{1}{R_2(x)} e\left(-j\frac{2\pi}{\lambda}R_2(x)\right)$$

For a small array, the field can be conveniently approximated as:

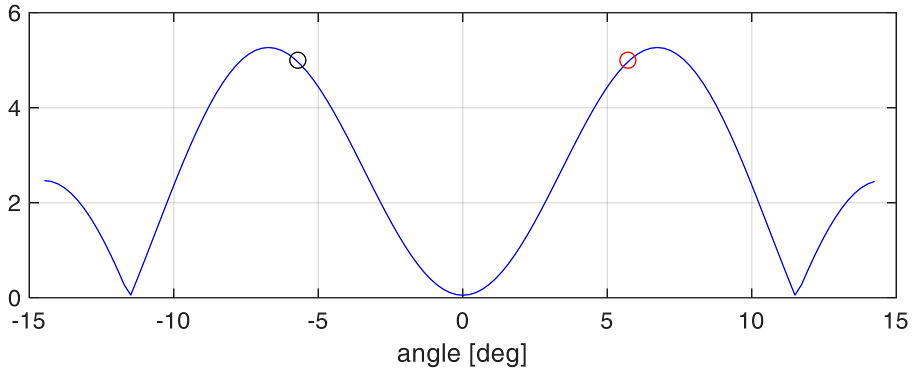
$$E(x) = \frac{1}{R_1} e\left(-j\frac{2\pi}{\lambda}R_1\right) e\left(-j\frac{2\pi}{\lambda}sin\theta_1 x\right) + \frac{1}{R_2} e\left(-j\frac{2\pi}{\lambda}R_2\right) e\left(-j\frac{2\pi}{\lambda}sin\theta_2 x\right)$$

With R_1 and R_2 denoting the distances of the two satellites to the center of the array (x=0). The two angles are: $\theta_1 = +5.76°$ and $\theta_2 = -5.76°$.

The angular separation is $\theta_1 - \theta_2 \approx 11°$ and the array angular resolution is $\Delta\theta \approx \frac{\lambda}{(5dx)} = 23°$. Accordingly, it is not possible to resolve the two satellites since their signals appear as a single peak (see graph below, the red and black points mark the true angular positions of the two satellites).



The situation changes if dx = 2 m, in which case the array angular resolution is $\Delta\theta \approx \frac{\lambda}{(5dx)} = 5.7°$. In this case it is possible to resolve the two satellites (see graph below).



The ambiguous spatial frequency is fa = 1/dx, so the ambiguous interval is $\left(-\frac{1}{2dx}, +\frac{1}{2dx}\right)$.

The ambiguous angle is obtained by the rule $f = \lambda sin(\theta)$, hence the ambiguous angular interval is (-90°,90°) for dx = 0.5 and (-14.5°,14.5°) for dx = 2.

Pseudo-code: essentially the same as in the first exercise.