Electromagnetics and Signal Processing for Spaceborne Applications
January 26th, 2024

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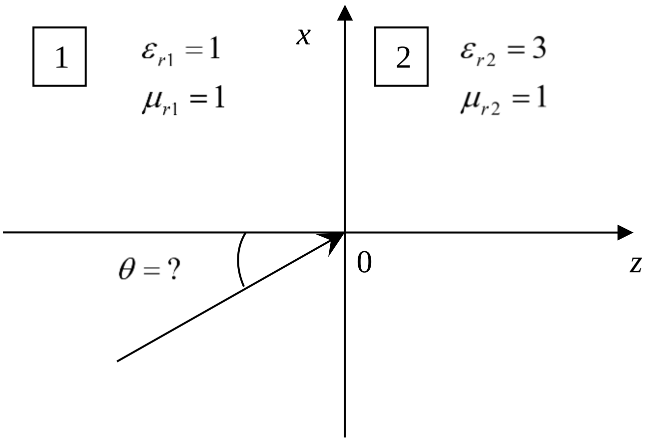
Problem 1

A sinusoidal plane wave at 1 GHz impinges on the discontinuity depicted in the figure below.



The incident wave is circularly polarized, it carries a power density S = 10 W/m^2, and the incident angle is the Brewster's angle.

Determine:

a) The incidence, reflection and refraction angles.
b) The TE and TM reflection coefficients at the interface.
c) The polarization of the reflected and transmitted wave.
d) The expression of the reflected electric field.

Solution:

a) Brewster angle (incidence and reflection angles):

$$\theta = \theta_B = \tan^{-1}\left(\frac{\sqrt{\varepsilon_{r2}\mu_{r2}}}{\sqrt{\varepsilon_{r1}\mu_{r1}}}\right) = 60°$$

The reflection angle is:

$$\theta_R = \theta = 60°$$

The refraction angle is given by:

$$n_1 \sin(\theta_B) = n_2 \sin(\theta_T) \quad \Rightarrow \quad \sin(\theta_T) = \sin(\theta_B)/\sqrt{3} \quad \Rightarrow \quad \theta_T = 30°$$

b) As the incidence angle is the Brester's angle → $\Gamma_{TM} = 0$
Regarding the TE component:

$$\eta_1^{TE} = \frac{\eta_0}{\cos(\theta_B)} = 754\ \Omega$$

$$\eta_2^{TE} = \frac{\eta_0}{\sqrt{3}\cos(\theta_T)} = 251.3\ \Omega$$

$$\Gamma_{TE} = \frac{\eta_2^{TE} - \eta_1^{TE}}{\eta_2^{TE} + \eta_1^{TE}} = -0.5$$

c) The reflected wave is only TE → linear polarization.
The polarization of the refracted wave → elliptical (different absolute values for the TE e TM components).

d) First it is necessary to define the incident TE component (at the interface). For example:
$E_i^1 = E_0\vec{\mu}_y$
The value of E_0 can be derived from the following expression:

$$S = \frac{1}{2}\frac{|\vec{E}_T|^2}{\eta_0} = \frac{1}{2}\frac{\left|\sqrt{E_0^2 + E_0^2}\right|^2}{\eta_0} = \frac{1}{2}\frac{2E_0^2}{\eta_0} = \frac{E_0^2}{\eta_0}$$

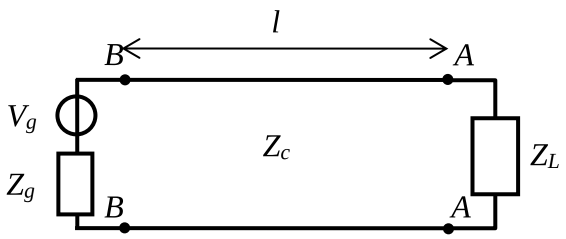
Therefore:
$E_0 = 61.4\ \mathrm{V/m}$
The reflected field is:
$E_r^1 = \Gamma_{TE}E_0\vec{\mu}_y e^{j\beta_1\cos(\theta)}e^{-j\beta_1\sin(\theta)}\ \ \mathrm{V/m}$

Problem 2

A source with voltage V_g = 50 V and internal impedance Z_g = 100 Ω is connected to a load Z_L = 150 Ω by a transmission line with characteristic impedance Z_C = 50 Ω. The line length is l = 4 m and the frequency is f = 300 MHz.

Calculate:

a) The power absorbed by the load
b) The voltage at the beginning of the line (section BB below), V_B
c) The trend of V_A in time



Solution

a) The wavelength is:

$$\lambda = \lambda_0 = c/f = 1 \text{ m}$$

The reflection coefficient at section AA is:

$$\Gamma_A = \frac{Z_L - Z_C}{Z_L + Z_C} = 0.5$$

The reflection coefficient at section BB is:

$$\Gamma_B = \Gamma_A e^{-j2\beta l} = \Gamma_A e^{-j2\left(\frac{2\pi}{\lambda}\right)4\lambda} = \Gamma_A e^{-j16\pi} = 0.5$$

Therefore, the input impedance is:

$$Z_B = Z_C \frac{1+\Gamma_B}{1+\Gamma_B} = 150 \ \ \Omega$$

The reflection coefficient for the source is:

$$\Gamma_g = \frac{Z_B - Z_g}{Z_B + Z_g} = 0.2$$

Therefore, the power crossing section BB, i.e. reaching the load is:

$$P_L = P_{AV}(1 - \left|\Gamma_g\right|^2) = 3 \text{ W}$$

b) The voltage at the beginning of the line is:

$$V_B = V_g \frac{Z_B}{Z_B + Z_g} = 30 \text{ V}$$

c) The progressive wave on the right side of section BB is:

$$V_B^+ = \frac{V_B}{1 + \Gamma_B} = 20 \text{ V}$$

The voltage at section AA is:

$$V_A = V_B^+ e^{-j\beta l}(1+\Gamma_A) = V_B^+ e^{-j\left(\frac{2\pi}{\lambda}\right)4\lambda}(1+\Gamma_A) = V_B^+ e^{-j8\pi}(1+\Gamma_A) = 30\ \mathrm{V}$$

The trend of V_A in time is given by:

$$v_A(t) = 30\cos(2\pi f t)\ \mathrm{V}$$

Problem 3

A ground penetrating Radar transmits an electromagnetic pulse with a total bandwidth of B=50 MHz centered about a carrier frequency f_0 = 200 MHz. The signal is reflected by two interfaces at a depth of 10 m and 15 m below the Radar. The transmitted signal is modeled as $s(t) = g(t)cos(2\pi f_0 t)$.

1. Write the expression of the complex transmitted signal, of the complex received signal, and of the complex received signal after demodulation (the complex envelope).
2. Write the expression of the complex received signal after pulse compression and draw its graph. Can you distinguish the two interfaces?
3. A discrete-time signal s_n is obtained by sampling $s(t)$ with sampling frequency $f_s = 300\,MHz$. Write the expressions of the Fourier Transform of s_n and draw a graph of the magnitude. Is it possible to recover the complex envelope from the sequence s_n ?

Solution

1) The complex-valued transmitted signal is $s(t) = g(t)exp(j2\pi f_0 t)$ (so that you go back to the real-valued signal by taking the real part).
The complex received signal is
$s_{rx\_RF}(t) = g(t-\tau_1)exp\big(j2\pi f_0(t-\tau_1)\big) + g(t-\tau_2)exp\big(j2\pi f_0(t-\tau_2)\big)$
With $\tau_i = \frac{2R_i}{c}$.
The complex envelope is obtained by multiplying by $exp(-j2\pi f_0 t)$, thus:

$$s_{rx}(t) = g(t-\tau_1)exp(-j2\pi f_0\tau_1) + g(t-\tau_2)exp(-j2\pi f_0\tau_2)$$

2) after cross-correlation

$$s_{rx}(t) = sinc\big((t-\tau_1)B\big)exp(-j2\pi f_0\tau_1) + sinc\big((t-\tau_2)B\big)exp(-j2\pi f_0\tau_2)$$

The signal bandwidth allows for a range resolution $\frac{c}{2B}$ = 3 m, so it is possible to resolve the two interfaces (the peaks do not overlap)

3) The Fourier Transform of the continuous time signal $s(t)$ is

$$S(f) = \frac{1}{2}G(f-f_0) + \frac{1}{2}G(f+f_0)$$

After sampling, the signal component $G(f+f_0)$ gets replicated in $G(f-f_1)$ with $f_1 = 100$ Mhz, whereas the signal component $G(f-f_0)$ gets replicated in $G(f+f_1)$. It is immediate to see that the two components do not overlap. Therefore, it is always possible to separate them and demodulate the signal correctly.

Problem 4

A signal $s(t)$ with total (two-sided) bandwidth B = 100 MHz arrives at two receivers following two distinct paths. The signals output at the two receivers are modeled as:

$$d_1(t) = s(t) + s(t - \tau_1)$$
$$d_2(t) = s(t) + s(t - \tau_2)$$

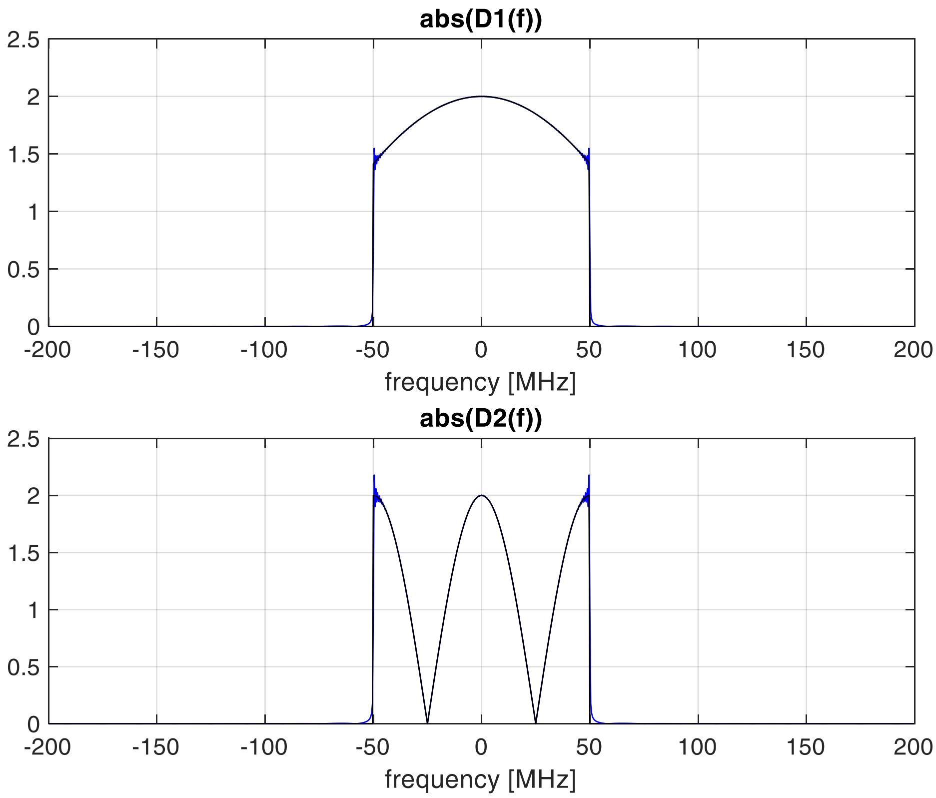
Where $\tau_1 =$ 5 nanoseconds and $\tau_2 =$ 20 nanoseconds.

1. Calculate the expression of the Fourier Transforms of $d_1(t)$ and $d_2(t)$ and draw the graphs of their absolute values, having care to highlight the positions where they are null (for this point, you can approximate *S*(*f*) to a rectangular pulse in the frequency domain).
2. Propose a procedure to restore s(t) using only $d_1(t)$. Is it possible to obtain s(t) with no errors?
3. Propose a procedure to restore s(t) using only $d_2(t)$. Is it possible to obtain s(t) with no errors?
4. Compute the cross-correlation between $d_1(t)$ and $d_2(t)$ and draw its graph.

Solution

$$D_1(t) = S(f)\big(1 + exp(-j2\pi f\tau_1)\big)$$
$$D_2(t) = S(f)\big(1 + exp(-j2\pi f\tau_2)\big)$$

The zeroes are found at frequencies for which $exp(-j2\pi f\tau) = -1$. In the case of d1 there's no zero within the signal bandwidth. In the case of d2 there are two zeroes at +-25 MHz, see the graphs below.



The procedure to restore the signal from d1 is to filter it by

$$H_1(t) = rectpuls(\frac{f}{B})\left(1 + exp(-j2\pi f\tau_1)\right)^{-1}$$

Where H1 can be inverted since it presents no zeroes.
This is not possible in the case of signal d2 due to the presence of zeroes. In this case one only could use an approximate inversion, for example

$$H_2(t) = rectpuls(\frac{f}{B})\left(1 + constant + exp(-j2\pi f\tau_2)\right)^{-1}$$

The cross-correlation is readily found as:

$$R_{21}(t) = d_2(t) * d_1^*(-t) = \left(s(t) + s(t-\tau_2)\right) * \left(s(t) + s(t-\tau_1)\right)^*_{|-t}$$

At this point one can simply note that each term of $R_{21}(t)$ produces the autocorrelation function of s(t) with a delay given by the difference between the delays of the first and the second signals.
Said $R_s(t)$ the autocorrelation of s(t), we have:

$$R_{21}(t) = R_s(t) + R_s(t-\tau_2) + R_s(t+\tau_1) + R_s(t-\tau_2+\tau_1)$$