FUNDAMENTALS OF ELECTROMAGNETIC FIELDS – July 18th, 2018

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Name and surname
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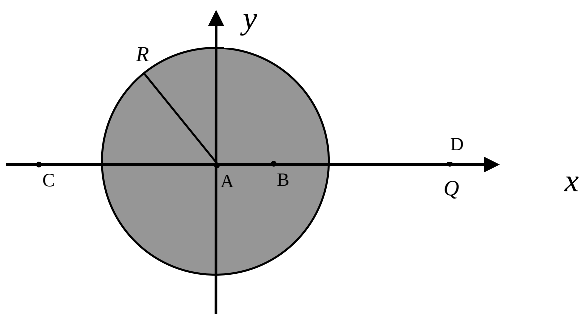
Problem 1 - A uniform charge density $\rho = 10^{-10}$ C/m^3 is embedded in a dielectric sphere with radius R = 4 m (grey part in the figure). A point charge, with value Q = - 3·10^{-9} C, is fixed in point D(2R,0). Calculate the value of the electric field in A(0,0), B(1/2R,0) and in C(-3/2R,0)



Solution

The calculation of the electric field relies on the use of the Gauss' theorem:

$$\int_S \vec{D}\cdot d\vec{S} = \int_V \rho\, dV$$

For A(0,0)

There is no contribution by the charged sphere (the integration domain zero), so the only contribution comes from the charge in D (which is negative):

$$\vec{E}_A = \frac{|Q|}{4\pi\varepsilon_0 (2R)^2}\,\vec{\mu}_x = 0.42\,\vec{\mu}_x \text{ V/m}$$

For B(0,0)

The contribution of the sphere can be found by solving the Gauss' theorem:

$$\varepsilon_0 \left|\vec{E}_B^{'}\right| 4\pi\left(x_B\right)^2 = \rho\frac{4}{3}\pi\left(x_B\right)^3 \quad \Rightarrow \quad \vec{E}_B^{'} = \frac{\rho}{3\varepsilon_0} x_B\,\vec{\mu}_x = \frac{\rho}{3\varepsilon_0} x_B\,\vec{\mu}_x = 7.53\vec{\mu}_x \text{ V/m}$$

The contribution of the charge in D is:

$$\vec{E}_B^{''} = \frac{|Q|}{4\pi\varepsilon_0 (2R - x_B)^2}\vec{\mu}_x = 0.75\vec{\mu}_x \text{ V/m}$$

So the total field is:

$$\vec{E}_B = \vec{E}_B^{'} + \vec{E}_B^{''} = 8.28\vec{\mu}_x \text{ V/m}$$

<u>For C(0,0)</u>

The contribution of the sphere can be found again by solving the Gauss' theorem:

$$\varepsilon_0 \left|\vec{E}_C^{'}\right| 4\pi\left(x_C\right)^2 = \rho\frac{4}{3}\pi\left(R\right)^3 = Q_T = 2.681\cdot 10^{-8}\ C \quad \Rightarrow \quad \vec{E}_C^{'} = -\frac{Q_T}{4\pi\varepsilon_0\left(x_C\right)^2}\vec{\mu}_x = -6.69\vec{\mu}_x$$

V/m

The contribution of the charge in D is:

$$\vec{E}_C^{''} = \frac{|Q|}{4\pi\varepsilon_0 (2R + x_C)^2}\vec{\mu}_x = 0.14\vec{\mu}_x \text{ V/m}$$

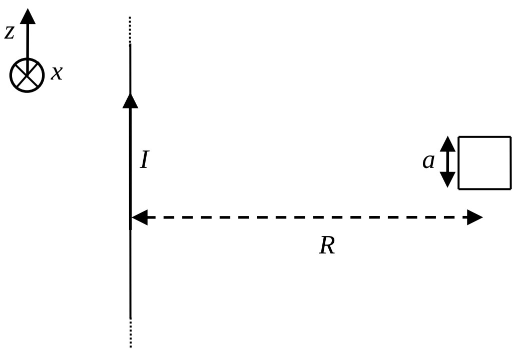
So the total field is:

$$\vec{E}_B = \vec{E}_B^{'} + \vec{E}_B^{''} = -6.55\vec{\mu}_x \text{ V/m}$$

<u>Problem 2</u> - Consider a square metallic ring with lateral dimension a = 0.01 m and the electric current I flowing along a straight metallic wire of indefinite length. The temporal trend of the current is:

$$I = \begin{cases} 0 & t < 0\,\mathrm{s} \\ t\,\vec{\mu}_z & 0 \le t < 3\,\mathrm{s} \\ 3\,\vec{\mu}_z & t \ge 3\,\mathrm{s} \end{cases} \quad \mathrm{A}$$

The distance between wire and the metallic ring is R = 1 m. Calculate the current flowing in the metallic ring, including its direction. Assume then that the metallic ring is associated to a resistance R = 5 Ω and that, given $R >> a$, the magnetic field generated by I can be considered to be constant for any point inside the metallic ring.



Solution

The magnetic field generated by the wire and flowing across the metallic ring is:

$$\vec{H} = \frac{I}{2\pi R}\vec{\mu}_x \quad \text{(A/m)}$$

The magnetic flux is given by (assuming the magnetic field is constant inside the ring):

$$\phi = \int_S \vec{B}\cdot dS = \mu_0 \left|\vec{H}\right| A = \mu_0 \left|\vec{H}\right| a^2 = \mu_0 \frac{I}{2\pi R} a^2 \quad \text{(Wb)}$$

Therefore the electromotive force is:

$$V(t) = \left| -\frac{\partial \phi}{\partial t} \right| = \begin{cases} 0 & t < 0\,\mathrm{s} \\ \dfrac{\mu_0 a^2}{2\pi R} = 2\cdot 10^{-11} & 0 \le t < 3\,\mathrm{s} \\ 0 & t \ge 3\,\mathrm{s} \end{cases} \quad \text{(V)}$$

Thus the current, for $0 \le t < 3\,\mathrm{s}$, is (zero otherwise):

$I(t) = V(t)/R =$ 4 pA

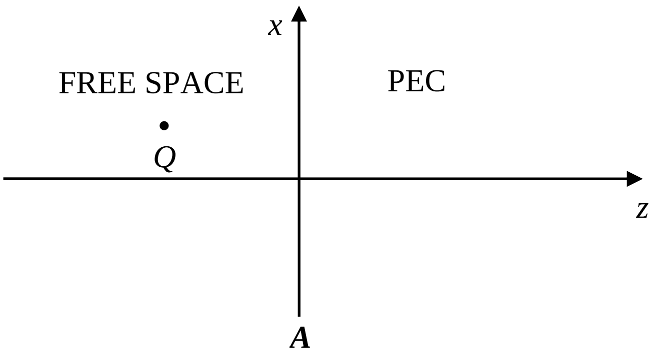
The direction of the current is counter-clockwise.

<u>Problem 3</u> - A uniform plane wave propagates in free space and impinges on a perfect electric conductor. The incident electric field is

$$\vec{E}_i = e^{-j20.944z}\ \vec{\mu}_y \ \text{V/m}$$

Calculate:

a) The frequency of the wave
b) The wavelength of the wave in the first medium
c) The electric field in Q(0.6,0.6,-0.6)



Solution

a) The information on the frequency of the wave is embedded in phase constant β = 20.944 rad/m. In free space its expression is:

$$\beta = \frac{2\pi f}{c}$$

Therefore f = 1 GHz.

b) In free space, the wavelength is given by:

$$\lambda = \frac{c}{f} = 0.3 \ \text{m}$$

c) The electric field in Q is a combination of the incident and of the reflected field. The second one is found by studying the reflection coefficient. For the discontinuity of the problem:

$$\Gamma = \frac{\eta_{PEC} - \eta_{FS}}{\eta_{PEC} + \eta_{FS}} = \frac{0-377}{0+377} = -1$$

In fact, as expected, since the wave cannot penetrate into the metal, the electric field is totally reflected, but with the opposite sign, such that at the discontinuity A, the total tangential electric field is zero.

The reflected field is therefore:

$$\vec{E}_r = -e^{j20.944z}\ \vec{\mu}_y \ \text{V/m}$$

The total electric field in Q is found by summing up both incident and reflected fields and by setting z = -0.6 (the other two coordinates are not important as the wave is a plane wave):

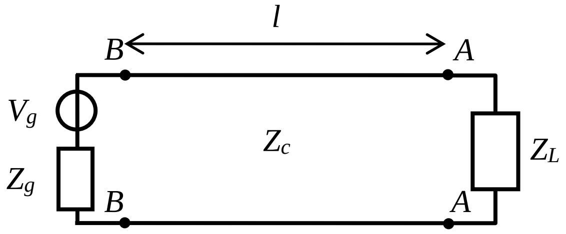
$$\vec{E}_T(Q) = \vec{E}_i(Q) + \vec{E}_R(Q) = \vec{\mu}_y - \vec{\mu}_y = 0 \text{ V/m}$$

In fact, 0.3 is a multiple of the wavelength, i.e. the result is the same as the one on the discontinuity.

Problem 4 - A source with voltage V_g = 10 V and internal impedance Z_g = 50 Ω is connected to a load Z_L = 50 Ω by a transmission line with characteristic impedance Z_C = 150 Ω. The line length is l = 6 m and the frequency is f = 400 MHz.

Calculate:

a) The power absorbed by the load
b) The temporal trend of the voltage at the beginning of the line (section BB below), V_B
c) The absolute value of the voltage at section AA below, V_A



Solution

a) The wavelength is:

$$\lambda = \lambda_0 = c/f = 0.75 \text{ m}$$

The reflection coefficient at section AA is:

$$\Gamma_A = \frac{Z_L - Z_C}{Z_L + Z_C} = -0.5$$

The reflection coefficient at section BB is:

$$\Gamma_B = \Gamma_A e^{-j2\beta l} = \Gamma_A e^{-j2\left(\frac{2\pi}{\lambda}\right)l} = -0.5$$

Therefore, the input impedance is:

$$Z_B = Z_C \frac{1+\Gamma_B}{1-\Gamma_B} = 50 \;\; \Omega = Z_L$$

There is perfect match at section BB, therefore

$$\Gamma_g = \frac{Z_B - Z_g}{Z_B + Z_g} = 0$$

As a result, all the power made available by the generator will cross section BB and reach the load:

$$P_L = P_{AV} = \frac{\left|V_g\right|^2}{8\,\mathrm{Re}\left(Z_g\right)} = 0.25 \text{ W}$$

b) The voltage at the beginning of the line is found as:

$$V_B = V_g \frac{Z_B}{Z_B + Z_g} = 5\ \mathrm{V}$$

Therefore, the trend of V_B in time is given by:

$$v_B(t) = \mathrm{Re}\left[V_B e^{j\omega t}\right] = 5\cos(\omega t)\ \ \mathrm{V}$$

c) The absolute value of the voltage at the load section can be obtained by inverting the following equation:

$$P_L = \frac{1}{2}\frac{|V_A|^2}{Z_L}$$

Thus:

$$|V_A| = \sqrt{2P_L Z_L} = 5 \quad \mathrm{V}$$